An (s,S)-inventory system with exponentially distributed lead times

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Research memorandum

TILBURG INSTITUTE OF ECONOMICS
DEPARTMENT OF ECONOMETRICS

An (s,S)-inventory system with
exponentially distributed lead times.

by

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Summary.

The paper discusses an inventory model with a single product and a single stocking point. The demands are independent random variables with common distribution F. The times between demands are independent random variables with common distribution G. Excess of demand is backlogged. An (s,S)-ordering policy is followed. The lead times of orders are independent random variables with common distribution H.

For arbitrary F, exponential G and H the limit distributions are derived for the stock on hand, the number of outstanding orders and the delivery time of the inventory system.

Contents.

1. Introduction.

We consider an inventory model with a single product and a single stocking point. The demands D_1, D_2,... are non-negative independent random variables with common distribution F, not degenerate at 0. The times between demands T_1, T_2,... are positive independent random variables with common distribution G. Excess of demand is backlogged until it is subsequently filled by a delivery.
The ordering policy followed is an (s,S)-policy. At the moment that the stock on hand plus on order equals or falls below s this stock level is ordered up to S ($s < S$). Otherwise no ordering is done. The lead times L_1, L_2,... of orders are positive independent random variables with common distribution H. It is supposed that the processes $\{D_n, n = 1,2,...\}$, $\{T_n, n = 1,2,...\}$ and $\{L_n, n = 1,2,...\}$ are mutually independent.

We are interested in the limit distribution of the stock on hand W(t), the stock on hand plus on order V(t) and the number of outstanding orders K(t) at time t for $t \to \infty$.

Problems of this kind have been studied in literature for various assumptions with respect to F, G and H. For some basic results, we refer to Arrow, Karlin, Scarf [1], Morse [5], Prabhu [6], and in particular to Karlin, Scarf [4] and Scarf [7].

For the case that F is degenerate at 1, G is non-lattice and H is degenerate at $c > 0$, the limit distribution of V(t), W(t) and K(t) can be found in Galliher, Morse, Simond [3]. An extension for arbitrary H is given in Finch [2], and an extension for arbitrary F restricted to the integers in Tijms [10].

For the case that F is degenerate at 1, G and H are exponential, the limit distribution of V(t), W(t) and K(t) can also be found in Galliher, Morse, Simond [3]. An extension for arbitrary G can be found again in Finch [2].

It seems that no attention has been paid to the case of arbitrary F and exponential G and H. In this paper we will treat this problem.

The basic result is the expression for the simultaneous limit distribution of V(t) and K(t) (section 2). This result enables us to determine the limit distribution of W(t) (section 3). In section 4 we determine the limit distribution of the delivery time L(t) of the inventory system, that is, the non-negative time elapsing from time t till the moment that the stock on hand becomes positive under the condition that after t no demands arrive. Proofs of the theorems can be found in section 5.

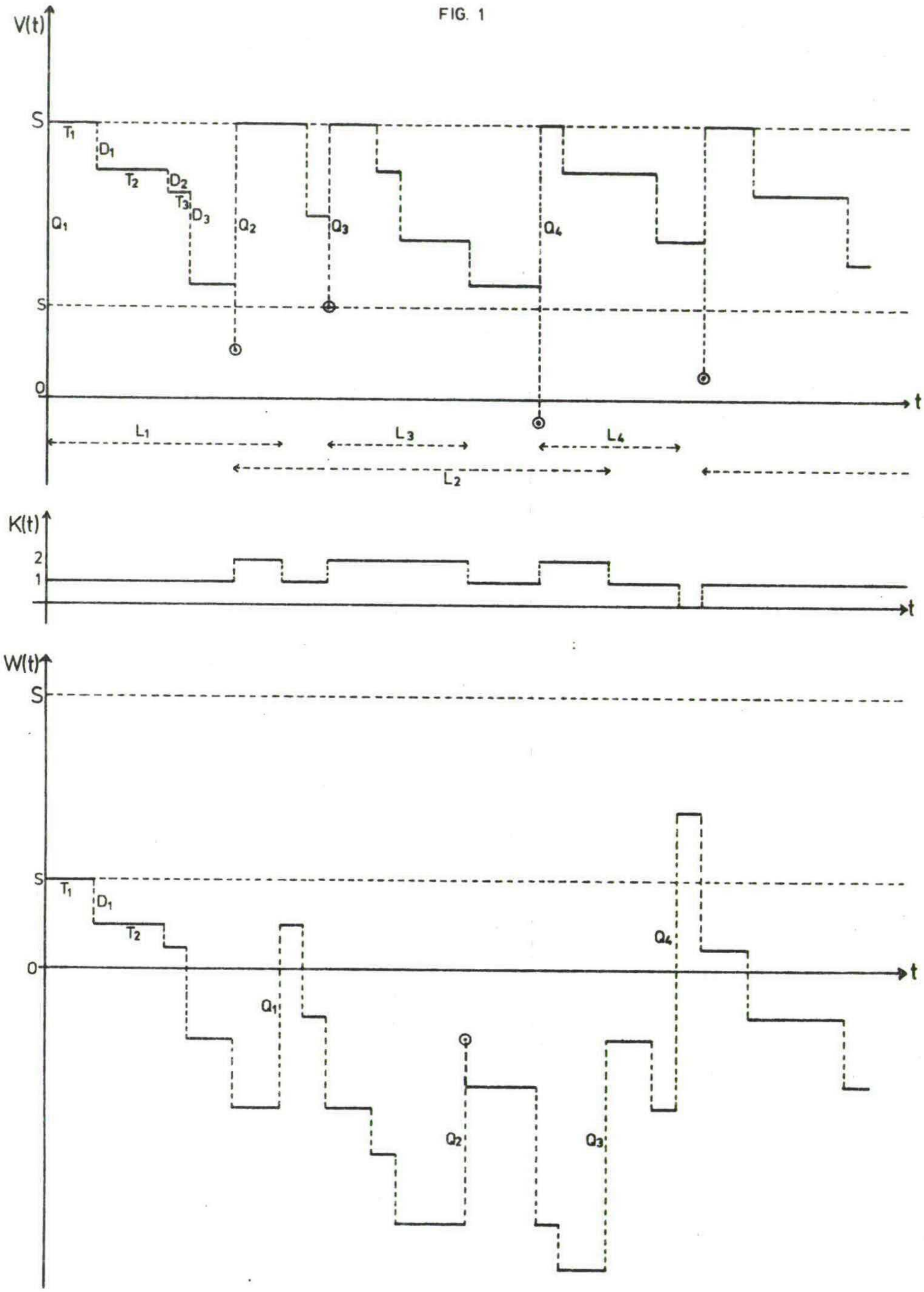
The foregoing description does not define the processes {V(t)}, {W(t)}, {K(t)} and {L(t)} in a unique way. To fix the ideas, we add the conditions that these processes are continuous from the right for all $t \geq 0$ with initiary conditions V(0) = S, W(0) = s, K(0) = 1 (see figure 1). So for the case $s > 0$ we have L(0) = 0. Of course, in general these conditions do not affect limit distributions.

In what follows we find it easier to work with the stock deficit on hand and on order X(t), and the stock deficit on hand Y(t), in stead of V(t) and W(t). These quantities are defined for all $t \geq 0$ by

$$X(t) = S - V(t), \; Y(t) = S - W(t).$$

All conclusions concerning X(t) and Y(t) can be immediately transformed to V(t) and W(t).

FIG. 1


V(t)
S
T1
D1
T2
D2
T3
D3
Q1
Q2
Q3
Q4
s
0
t
L1
L3
L4
L2
K(t)
2
1
t
W(t)
S
s
T1
D1
T2
0
t
Q1
Q2
Q3
Q4

2. The limit distribution of $\{X(t), K(t)\}$.

With respect to the distribution F we write $F\{A\}$ for the probability mass of the set $A \subset (-\infty,\infty)$ and $F(x)$ for the value of the distribution function at $x \in (-\infty,\infty)$ (taken continuously from the left). Similar conventions apply to other distributions.

Expressions for infinitesimal sets $\{dx\}$ at x should be interpreted as integral expressions.

Since G and H are exponential we can write

$$(2.1) \qquad \begin{cases} G(x) = 1 - e^{-\lambda x} & , x \geq 0 \\ H(x) = 1 - e^{-\mu x} & , x \geq 0 \end{cases}$$

with λ, $\mu > 0$.

Let

$$(2.2) \qquad \zeta = S - s \quad , \quad \alpha = \lambda/\mu,$$

and for any $A \subset (-\infty,\infty)$ and integer k:

$$(2.3) \qquad P_t\{A,k\} = P\{X(t) \in A,\ K(t) = k\}.$$

The process $\{X(t), K(t), t \geq 0\}$ is a Markov-process with stationary transition probabilities. We have:

Theorem 1. The forward-equations of the Markov-process $\{X(t), K(t), t \geq 0\}$ with stationary transition probabilities are given by

$$(2.4) \qquad \frac{\partial}{\partial t} P_t\{dx,k\} + (\lambda+k\mu).P_t\{dx,k\} =$$

$$= (k+1)\mu.P_t\{dx,k+1\} + \lambda \int_{[0,x)} F\{dx-y\}P_t\{dy,k\},$$

$$\text{if } 0 < x < \zeta,\ k = 0,1,\ldots,$$

(2.5) $$\frac{\partial}{\partial t} P_t\{0,k\} + (\lambda+k\mu).P_t\{0,k\} =$$

$$= (k+1)\mu.P_t\{0,k+1\} + \lambda \int_{[0,\zeta)} \{1-F(\zeta-y)\}P_t\{dy,k-1\}$$

if $k = 0,1,\ldots$ (and $x = 0$).

Proof. see section 5.

Suppose that $\{X(t), K(t)\}$ converges for $t \to \infty$ in distribution to (X, K). Let

(2.6) $$P_k\{A\} = P\{X \in A, K = k\}$$

for $A \subset (-\infty,\infty)$ and integer k.
Since $P_t\{dx,k\} \to P_k\{dx\}$ for $t \to \infty$ we get from (2.2), (2.4), (2.5) by omitting derivatives:

(2.7) $$(\alpha+k)P_k\{dx\} = (k+1).P_{k+1}\{dx\}+\alpha \int_{[0,x)} F\{dx-y\}P_k\{dy\}$$

if $0 < x < \zeta$, $k = 0,1,\ldots,$

(2.8) $$(\alpha+k)P_k\{0\} = (k+1).P_{k+1}\{0\}+\alpha \int_{[0,\zeta)} \{1-F(\zeta-y)\}P_{k-1}\{dy\}$$

if $k = 0,1,\ldots$ (and $x = 0$).

The relations (2.7), (2.8), considered as an equation-system in $P_k\{A\}$ for $A \subset (-\infty,\infty)$ and $k = 0,1,\ldots$ (with the convention $P_{-1} \equiv 0$) are the stationary equations of the Markov-process $\{X(t), K(t)\}$. If sufficiently many sets of states of this process communicate with each other the (probability) solution of the stationary equations is uniquely determined, does not depend on the initiary conditions and provides the limit distribution. In what follows we suppose that this is the case (e.g. for integer S and s, and lattice

F with span 1 this is guaranteed).

The following theorem presents the form of the limit distribution. We introduce the (generalized) renewal function belonging to F:

$$(2.9)\qquad \begin{cases} U(c,x) = \sum_{k=0}^{\infty} c^k F^{(k*)}(x), & 0 \leq c \leq 1, \\ U(x) = U(1,x) & , \end{cases}$$

for $x \geq 0$. Here $F^{(k*)}$ denotes the k-fold convolution of F with itself, and $F^{(0*)}$ has to be interpreted as the distribution degenerate at 0.

Theorem 2. The process $\{X(t),\ K(t)\}$ converges for $t \to \infty$ in distribution to (X,K) with

$$(2.10)\qquad P_k(x) = P\{X < x,\ K = k\} =$$

$$= \sum_{n=k}^{\infty} (-1)^{n-k} \binom{n}{k} E\{\binom{K}{n}\} U(\tfrac{\alpha}{n+\alpha}, x)/U(\tfrac{\alpha}{n+\alpha}, \zeta)$$

$$\text{if } 0 < x \leq \zeta,\ k = 0,1,\ldots.$$

The binomial moments of K are determined by the recurrence relation

$$(2.11)\qquad E\{\binom{K}{n+1}\} = \frac{1}{n+1}\{(n+\alpha)/U(\tfrac{\alpha}{n+\alpha},\zeta)-n\}.E\{\binom{K}{n}\},\ n = 0,1,\ldots$$

with $E\{\binom{K}{0}\} = 1$.

In particular, for the marginal distributions we have

$$(2.12)\qquad q_k = P\{K = k\} = \sum_{n=k}^{\infty} (-1)^{n-k} \binom{n}{k} E\{\binom{K}{n}\},\ k = 0,1,\ldots,$$

$$(2.13)\qquad P(x) = P\{X < x\} = U(x)/U(\zeta),\ 0 < x \leq \zeta.$$

Proof. see section 5.

Remark. The relation (2.12) is not a special property of the distribution of K. According to Jordan's theorem it holds for any distribution which is restricted to the non-negative integers and completely determined by its moments (see e.g. Takács [9]).

The relations (2.11) and (2.13) cannot considered to be quite new. In fact, (2.13) gives the limit distribution of the stock on hand if the delivery of orders is immediate (compare Prabhu [6]). The relation (2.11) can be obtained from the result of Takács [8], Ch. 3, §3, for the limit distribution of the number of custumors in an $G/M/\infty$-queue by choosing a particular G. However, the relation (2.10) is new, and it is exactly this expression which enables us to derive the limit distribution of the stock deficit on hand $Y(t)$ for $t \to \infty$.

The following theorem gives a useful approximation of the tail of the distribution of (X,K) with respect to K.

Theorem 3. If $F(\zeta) < 1$ then

$$\lim_{k\to\infty} \frac{P\{X < x,\ K = k\}}{P\{Z = k\}} = 1, \quad 0 < x \leq \zeta,$$

where Z has a Poisson-distribution with mean $\alpha\{1 - F(\zeta)\}$.

Proof. see section 5.

The foregoing theorem can be interpreted as follows. For large k we may expect that the stock deficit $X(t)$ is close to 0. Then the arrival process of orders has an intensity which can be approximated by $\lambda\{1 - F(\zeta)\}$, the product of the intensity λ of arriving demands and the probability $1 - F(\zeta)$ that an order arrives. The intensity of the departure process of orders equals μ. A standard result for an $M/M/\infty$-queue gives that the limit distribution of the number of orders is the Poisson distribution with mean $\lambda\{1 - F(\zeta)\}/\mu = \alpha\{1 - F(\zeta)\}$.

3. The limit distribution of $\{Y(t)\}$.

Let $Q_1, Q_2, \ldots$ denote the sequence of successive orders (see figure 1). The distribution of $Y(t)$ is equal to that of

$$X(t) + \sum_{j=1}^{K(t)} Q_j ,$$

where the Q_j are independent and identically distributed and also independent of $(X(t), K(t))$. Therefore $\{X(t), Y(t), K(t)\}$ converges for $t \to \infty$ in distribution to (X, Y, K) with Y defined by

$$(3.1) \qquad Y = X + \sum_{j=1}^{K} Q_j ,$$

where (X, K) and $\{Q_1, Q_2, \ldots\}$ are independent.

Let Q have the distribution of the Q_j, and denote by R the distribution of $Q - \zeta$. Then R can be interpreted as the distribution of the residual life at time ζ in a renewal process generated by $D_1, D_2, \ldots$ with distribution F. In renewal theory the expression in terms of F and U is standard:

$$(3.2) \qquad R(t) = P\{Q - \zeta < t\} =$$

$$= \int_{[0,t)} \{U(\zeta)-U(\zeta-y)\}F\{dy\} + [U(\zeta)-U((\zeta-t)^+)]\{1-F(t)\}$$

for all $\zeta, t > 0$.

The following theorem gives the distribution function of Y. Here an empty sum has to be interpreted as zero.

<u>Theorem 4.</u> The process $\{Y(t)\}$ converges for $t \to \infty$ in distribution to Y with

$$
(3.3) \quad \begin{cases} P\{Y < y\} = P_0(y) \ , \ 0 < y \le \zeta, \\ P\{Y < y\} = q_0 + \sum_{k=1}^{m-1} \int_{[0,\zeta)} R^{(k*)}(y-x-k\zeta)P_k\{dx\} + \\ \qquad + \int_{[0,y-m\zeta)} R^{(m*)}(y-x-m\zeta)P_m\{dx\} \\ \text{if } m\zeta < y \le (m+1)\zeta, \ m = 1,2,\ldots. \end{cases}
$$

<u>Proof.</u> see section 5.

From (3.1) some interesting relationships between moments can be derived. We only state

$$(3.4) \qquad E\{Y\} = E\{X\} + E\{K\}.E\{\Omega\}.$$

We proceed with the particular case that F is exponential:

$$(3.5) \qquad F(x) = 1 - e^{-\nu x}, \ x \ge 0,$$

with $\nu > 0$. Then from (2.9) it follows that

$$U(c,x) = \frac{1}{1-c}\{1 - ce^{-\nu(1-c)x}\}, \ 0 \le c < 1,$$

$$U(1,x) = U(x) = 1 + \nu x,$$

and from (3.2) or the lack of memory of the exponential distribution that $R(x) = F(x)$, $x \ge 0$.
Set

$$(3.6) \qquad \beta = \nu\zeta.$$

Then it follows that

$$(3.7) \qquad E\{\Omega/\zeta\} = \frac{1+\beta}{\beta} \ ,$$

and with (2.12) and (2.13) that

(3.8) $\quad E\{X/\zeta\} = \frac{1}{2}\,\frac{\beta}{1+\beta}, \; E\{K\} = \frac{\alpha}{1+\beta}.$

Combining (3.4), (3.7) and (3.8) we get the nice relationship

(3.9) $\quad E\{Y/\zeta\} = E\{X/\zeta\} + \alpha/\beta.$

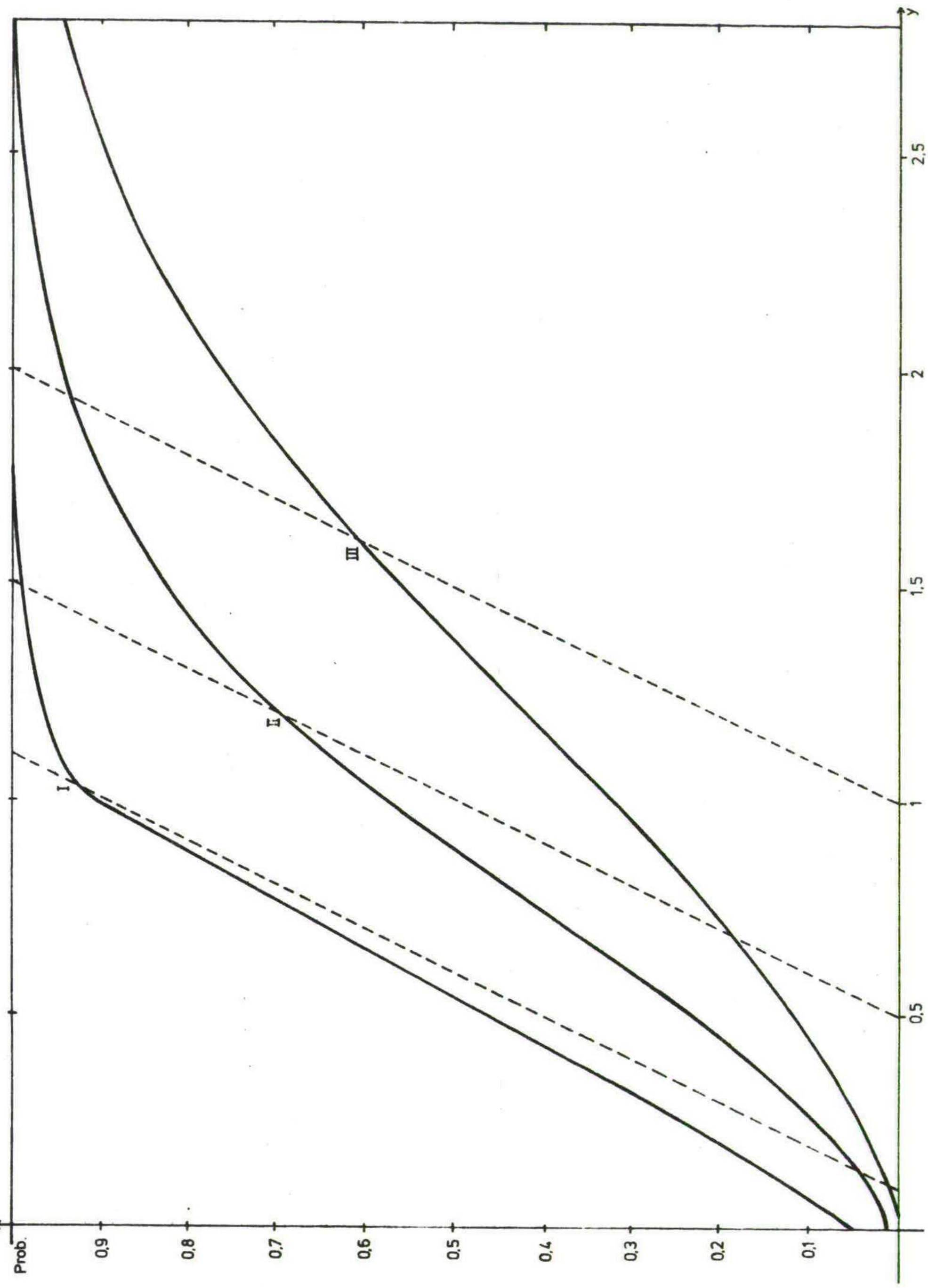
Although considerable simplification can be obtained for exponential F the formula in (3.3) for the distribution function of Y remains rather inattractive. We omit this formula and only present numerical results for some values of (α,β). These values are given below.

	α	β	σ	ρ
I	1	10	0.36	0.55
II	5	10	0.59	0.09
III	10	10	0.82	0.02

In this table σ stands for the standard deviation of Y/ζ and ρ for the correlation coefficient of X and Y.

Figure 2 illustrates the distribution of Y/ζ for these values. Here I-III denote the graphs of the distribution functions of Y/ζ for the indicated values of (α,β). The dotted lines I-III have been added for comparison. They correspond to the distribution functions of the stock deficit on hand (in units ζ) at a random epoch in a completely deterministic inventory model, for which the demand per unit of time and the delivery time are constants respectively equal to λ/ν and $1/\mu$. It must be noted that in the deterministic case the results only depend on α/β. However, in the random case the results depend on the separate values of α and β.

FIG. 2



4. The limit distribution of L(t).

We consider the limit distribution of $L(t)$ for $t \to \infty$ for the case that $s > 0$.

Let $L_1^{(k)},\ldots,L_k^{(k)}$, $k = 1,2,\ldots$, be the order statistics belonging to the first k random variables in the sequence $L_1, L_2,\ldots$. Then $L_j^{(k)}$, $j = 1,\ldots,k$, has the density

$$(4.1)\qquad \phi_j^{(k)}(t) = k\binom{k-1}{j-1}(1-e^{-\mu t})^{j-1}e^{-(k-j)\mu t}.\mu e^{-\mu t},\ t \geq 0.$$

We have:

Theorem 5. The process $\{L(t)\}$ converges for $t \to \infty$ in distribution to L, where the distribution of L has probability mass $P\{Y < s + \zeta\}$ at 0 and a density ψ on $(0,\infty)$, given by

$$(4.2)\qquad \psi(\tau) =$$

$$= \sum_{k=1}^{\infty}\sum_{j=1}^{k}\phi_{k-j+1}^{(k)}(\tau)\int_{[0,\zeta)}\left[\int_{I_j(x)}\{1-R(s-x-y-(j-1)\zeta)\}R^{((j-1)*)}\{dy\}\right]P_k\{dx\}$$

with

$$I_j(x) = [\,s-x-(j-1)\zeta,\ s-x-(j-2)\zeta).$$

Proof. see section 5.

For exponential F the formula in (4.2) can be simplified. However, the resulting form remains still unattractive. Therefore we omit this formula.

5. Proofs of the theorems.

The proof of theorem 1:

We write $X(t) = dx$ instead of $X(t) \in \{dx\}$. Consider the possible transitions of the process in $(t, t+h)$ to $X(t+h) = x$, $K(t+h) = k$ for all $0 \leq x < \zeta$, $k = 0,1,\ldots$. There are only three transitions which have a probability larger than $o(h)$ for $h \to 0$:

1°) No demands and no delivery. Starts at t with $X(t) = dx$, $K(t) = k$ and has probability

$$e^{-\lambda h}.e^{-k\mu h} = 1 - (\lambda+k\mu)h + o(h).$$

2°) No demands and one delivery. Starts at t with $X(t) = dx$, $K(t) = k+1$ and has probability

$$e^{-\lambda h}.(k+1)(1-e^{-\mu h})e^{-k\mu h} = (k+1)\mu h + o(h).$$

3°) One demand and no delivery. Starts at t for $0 < x < \zeta$ with $X(t) = dy$, $K(t) = k$ for some $0 \leq y < x$ and has conditional probability

$$(1-e^{-\lambda h}).F\{dx-y\} = \lambda h.F\{dx-y\} + o(h).$$

Starts at t for $x = 0$ and $k = 1,2,\ldots$ with $X(t) = dy$, $K(t) = k-1$ for some $0 \leq y < \zeta$ and has conditional probability

$$(1-e^{-\lambda h})\{1-F(\zeta-y)\} = \lambda h\{1-F(\zeta-y)\} + o(h),$$

while a transition for $x = 0$ and $k = 0$ is not possible. The results in 1°)-3°) lead for $0 < x < \zeta$, $k = 0,1,\ldots$ to

$$P\{X(t+h) = dx,\ K(t) = k\} =$$

$$= \{1-(\lambda+k\mu)h\}.P\{X(t) = dx,\ K(t) = k\} +$$

$$+ (k+1)\mu h.P\{X(t) = dx,\ K(t) = k+1\} +$$

$$+ \lambda h. \int_{[0,x)} F\{dx-y\}dP_y\{X(t) = dy,\ K(t) = k\} + o(h),$$

which gives (2.4). For $x = 0$, $k = 0,1,\ldots$ we get

$$P\{X(t+h) = 0,\ K(t) = k\} =$$

$$= \{1-(\lambda+k\mu)h\}.P\{X(t) = 0,\ K(t) = k\} +$$

$$+ (k+1)\mu h.P\{X(t) = 0,\ K(t) = k+1\} +$$

$$+ \lambda h \int_{[0,\zeta)} \{1-F(\zeta-y)\}dP_y\{X(t) = dy,\ K(t) = k-1\} + o(h),$$

since for $k = 0$ the integral in this equation vanishes. This gives (2.5) and completes the proof.

The proof of theorem 2.

Suppose there exists a solution for which $\sum_0^\infty q_k z^k$ has a radius of convergence $\rho > 1$.
Set

$$P\{z,dx\} = \sum_{k=0}^{\infty} z^k P_k\{dx\},\ |z| < \rho,\ 0 \le x < \zeta.$$

From (2.7) we get for all $0 < x < \zeta$ and $|z| < \rho$:

$$\text{(5.1)} \qquad \{\alpha-(1-z)\frac{\partial}{\partial z}\}P\{z,dx\} = \alpha \int_{[0,x)} F\{dx-y\}.P\{z,dy\},$$

and from (2.8) for $x = 0$ and all $|z| < \rho$, using $P_{-1} \equiv 0$, that

$$\text{(5.2)} \qquad \{\alpha-(1-z)\frac{\partial}{\partial z}\}P\{z,0\} = \alpha z \int_{[0,\zeta)} \{1-F(\zeta-y)\}P\{z,dy\}.$$

Integration of (5.1) leads to

$$\{\alpha-(1-z)\frac{\partial}{\partial z}\}[P(z,x)-P\{z,0\}] = \alpha\{F(x)*P(z,x)\}$$

for all $0 < x \leq \zeta$. Here $F(x)*P(z,x)$ stands for the value of the convolution of the distribution functions $F(\cdot)$ and $P(z,\cdot)$ at x.
With (5.2) this leads to

$$(5.3) \qquad \{\alpha-(1-z)\frac{\partial}{\partial z}\}P(z,x) =$$

$$= \alpha z[P(z,\zeta)-F(\zeta)*P(z,\zeta)] + \alpha F(x)*P(z,x)$$

for all $0 < x \leq \zeta$ and $|z| < \rho$.
Taking $x = \zeta$ in (5.3) we see that a factor $1-z$ cancels out. We get

$$(5.4) \qquad (\alpha-\frac{\partial}{\partial z})P(z,\zeta) = \alpha F(\zeta)*P(z,\zeta).$$

Combining (5.3) and (5.4) we get for all $0 < x \leq \zeta$ and $|z| < \rho$:

$$(5.5) \qquad \alpha P(z,x)-(1-z)\frac{\partial}{\partial z}P(z,x) = z\frac{\partial}{\partial z}P(z,\zeta)+\alpha F(x)*P(z,x).$$

We consider (5.5) as an equation in x and z for all $x > 0$ and $|z| < \rho$. Then a solution of this equation is a solution for the original problem for $0 < x \leq \zeta$ and $|z| < \rho$ provided that $P(1,\zeta) = 1$. The equation (5.5) admits a solution by means of Laplace transforms.
Let

$$f(s) = \int_{[0,\infty)} e^{-sx}F\{dx\},$$

$$p(z,s) = \int_{[0,\infty)} e^{-sx}P\{z,dx\},$$

for $s \geq 0$, and denote by $p^{(n)}$, $P^{(n)}$ the n^{th} partial derivative of p, P with respect to z, $n = 0,1,\ldots$. Then from (5.5) we get

$$\alpha\{1-f(s)\}.p(z,s) = (1-z)p^{(1)}(z,s) + zP^{(1)}(z,\zeta).$$

Differentiation at $z = 1$ of this relation gives

$$(5.6) \qquad [n+\alpha\{1-f(s)\}]p^{(n)}(1,s) = P^{(n+1)}(1,\zeta) + nP^{(n)}(1,\zeta)$$

for all $n = 0,1,2,\ldots$. Since

$$[n+\alpha\{1-f(s)\}]^{-1} = \frac{1}{n+\alpha} \sum_{k=0}^{\infty} \left(\frac{\alpha}{n+\alpha}\right)^k \{f(s)\}^k,$$

we get from (5.6) that

$$(5.7) \qquad p^{(n)}(1,x) = \frac{P^{(n+1)}(1,\zeta)+nP^{(n)}(1,\zeta)}{n+\alpha}.U\left(\frac{\alpha}{n+\alpha},x\right)$$

for all $x > 0$, $n = 0,1,2,\ldots$. By taking $x = \zeta$ we deduce from (5.7) that

$$(5.8) \qquad P^{(n)}(1,x) = P^{(n)}(1,\zeta).U\left(\frac{\alpha}{n+\alpha},x\right)/U\left(\frac{\alpha}{n+\alpha},\zeta\right).$$

For $n = 0$ with $P(1,\zeta) = 1$ this gives (2.13).
Now,

$$(5.9) \qquad P^{(n)}(1,\zeta) = \sum_{j=n}^{\infty} j(j-1)\ldots(j-n+1)q_k =$$

$$= E\{K(K-1)\ldots(K-n+1)\} = n!\ E\{\tbinom{K}{n}\}.$$

With (5.7) for $x = \zeta$ and $P(1,\zeta) = 1$ this leads to (2.11).
Furthermore,

$$P(z,x) = \sum_{n=0}^{\infty} \frac{(z-1)^n}{n!} P^{(n)}(1,x) =$$

$$= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \binom{n}{k} z^k (-1)^{n-k} \frac{1}{n!} P^{(n)}(1,x) =$$

$$= \sum_{k=0}^{\infty} \left[\sum_{n=k}^{\infty} (-1)^{n-k} \binom{n}{k} \cdot \frac{1}{n!} P^{(n)}(1,x) \right] z^k,$$

or

$$P_k(x) = \sum_{n=k}^{\infty} (-1)^{n-k} \binom{n}{k} \frac{1}{n!} P^{(n)}(1,x).$$

With (5.8), (5.9) this gives (2.10). The relation (2.11) follows from (2.10) by taking $x = \zeta$.

This gives the proof, provided that we can show that the foregoing solution is such that $\sum_0^{\infty} q_k z^k$ has a radius of convergence $\rho > 1$.

We will show that $\rho = \infty$.

Let for $0 < x \leq \zeta$:

$$\gamma_n(x) = \left[n!\ E\{\binom{K}{n}\} U\left(\frac{\alpha}{n+\alpha}, x\right) / U\left(\frac{\alpha}{n+\alpha}, \zeta\right) \right]^{\frac{1}{n}}.$$

Then with (2.10) and (2.11) we get

$$(5.10) \qquad P_k(x) = \sum_{n=k}^{\infty} (-1)^{n-k} \frac{\{\gamma_n(x)\}^n}{k!\ (n-k)!} = \frac{1}{k!} \sum_{n=0}^{\infty} (-1)^n \frac{\{\gamma_{n+k}(x)\}^n}{n!}$$

This gives that $\rho = \infty$ if $\gamma_n(\zeta)$ is bounded in n. In fact we have

$$(5.11) \qquad \lim_{n\to\infty} \gamma_n(x) = \alpha\{1-F(\zeta)\}, \ 0 < x \leq \zeta.$$

This can be shown as follows. From (2.11) we get

$$\lim_{n\to\infty} \gamma_n(x) = \lim_{n\to\infty} \left[\frac{U\left(\frac{\alpha}{n+\alpha}, x\right)}{U\left(\frac{\alpha}{n+\alpha}, \zeta\right)} \prod_{j=0}^{n} \left\{ \frac{j+\alpha}{U\left(\frac{\alpha}{j+\alpha}, \zeta\right)} - j \right\} \right]^{\frac{1}{n}} =$$

$$= \lim_{n\to\infty} \exp\left[\frac{1}{n} \sum_{j=0}^{n} \log\left\{\frac{j+\alpha}{U\left(\frac{\alpha}{j+\alpha},\zeta\right)} - j\right\}\right] =$$

$$= \lim_{n\to\infty} \left[(n+\alpha)/U\left(\frac{\alpha}{n+\alpha},\zeta\right) - n\right] =$$

$$= \lim_{n\to\infty} \frac{\alpha - n \sum_{k=1}^{\infty} \left(\frac{\alpha}{n+\alpha}\right)^k F^{(k*)}(\zeta)}{\sum_{k=0}^{\infty} \left(\frac{\alpha}{n+\alpha}\right)^k F^{(k*)}(\zeta)} =$$

$$= \alpha\{1-F(\zeta)\}.$$

This completes the proof.

The proof of theorem 3.

The result immediately follows from the relations (5.10) and (5.11) in the proof of theorem 2.

The proof of theorem 4.

We write $Y = dy$ instead of $Y \in \{dy\}$.
From (3.1) we get

$$P\{Y = dy\} = P\left\{X + \sum_{j=1}^{k} Q_j = dy\right\} =$$

$$= \sum_{k=0}^{\infty} \int_{[0,\zeta)} P\left\{\sum_{j=1}^{k} Q_j = dy-x\right\} P_k\{dx\} =$$

$$= \sum_{k=0}^{\infty} \int_{[0,\zeta)} P\left\{\sum_{j=1}^{k} (Q_j-\zeta) = dy-x-k\zeta\right\}.P_k\{dx\} =$$

$$= \sum_{k=0}^{\infty} \int_{[0,\zeta)} R^{(k*)}\{dy-x-k\zeta\}.P_k\{dx\}.$$

Integration with respect to y gives

$$P\{Y < y\} = \sum_{k=0}^{\infty} \int_{[0,\zeta)} R^{(k*)}(y-x-k\zeta).P_k\{dx\}.$$

This leads to (3.3) since P_k is restricted to $[0,\zeta)$ and $R^{(k*)}$ to $[0,\infty)$.

The proof of theorem 5.

The process $\{L(t)\}$ converges for $t \to \infty$ in distribution to L, where L is defined as follows:

$$L = 0, \text{ if } Y < s + \zeta$$

and for $\tau > 0$:

$$L = \tau, \text{ if } K = k,\ Y - \sum_{i=1}^{j-1} Q_i \geq s+\zeta,\ Y - \sum_{i=1}^{j} Q_i < s+\zeta,\ L_j^{(k)} = \tau,$$

$$\text{for some } j = 1,\dots,k.$$

Thus L has probability mass $P\{Y < s+\zeta\}$ at 0 and some density ψ.
Under the condition $\{X = x, K = k\}$ the relation (3.1) gives

$$Y = x + \sum_{i=1}^{k} Q_i.$$

This leads for $\tau > 0$ to

$$L = \tau, \text{ if } x + \sum_{i=j}^{k} Q_i \geq s+\zeta,\ x + \sum_{i=j+1}^{k} Q_i < s+\zeta,\ L_j^{(k)} = \tau$$

$$\text{for some } j = 1,\dots,k.$$

Therefore,

(5.12) $\quad \psi(\tau) =$

$$= \sum_{k=1}^{\infty} \int_{[0,\zeta)} \left[\sum_{j=1}^{k} P\left\{ \sum_{i=j}^{k} Q_i \geq s+\zeta-x, \quad \sum_{i=j+1}^{k} Q_i < s+\zeta-x \right\} \right] \phi_j^{(k)}(\tau) P_k\{dx\}.$$

Now,

$$P\left\{ \sum_{i=j}^{k} Q_i \geq s+\zeta-x, \quad \sum_{i=j+1}^{k} Q_i < s+\zeta-x \right\} =$$

$$= P\left\{ \sum_{i=j+1}^{k} (Q_i-\zeta) < s-x-(k-j-1\zeta, \quad \sum_{i=j+1}^{k} (Q_i-\zeta)+(Q_j-\zeta) \geq s-x-(k-j)\zeta \right\}$$

$$= \int_{I_{k-j+1}(x)} \{1-R(s-x-(k-j)\zeta-y)\} R^{((k-j)*)}\{dy\}.$$

Substitution of this expression in (5.12) and replacing j by k-j+1 leads to (4.2).
This completes the proof.

G.W.

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